Singularities in the Boussinesq equation and in the generalized Korteweg–de Vries equation

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In this paper, two kinds of analytic singular solutions (finite-time and infinite-time singular solutions) of two classical wave equations (the Boussinesq equation and a generalized Korteweg–de Vries equation) are obtained by means of the improved homogeneous balance method and a nonlinear transformation. The solutions show that special singular wave patterns exist in the classical models of shallow water wave problem. [S1063-651X(00)03510-8]

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I. INTRODUCTION

All kinds of wave patterns [1], from linear systems to nonlinear systems, contribute to understanding complex wave phenomena. Generally, the wave patterns are symmetric, resulting from the fascinating effects of instability. When the instabilities of a nonlinear system are violent, the energy may focus into a spike at a point where special wave patterns (singularity in the nonlinear system) appear; the special wave patterns are local divergences in the amplitude or gradients of some physical field. As is known, a lot of spatiotemporal systems exhibit this phenomena, e.g., in the turbulence problem it is observed in Ref. [2] that large fluctuations in the derivatives occur and the dissipation field appears multifractal (a set of nested singularities); in the surface wave problem, the singularity formation has been observed and studied [3,4]: nonlinear optical systems exhibit self-focusing effects, which may lead to the collapse of the optical power density into local divergences that may have important consequences on the integrity of optical fibers and laser systems: etc.

For understanding singularities of nonlinear physical systems, it is helpful to study singular solutions of partial differential equations (PDEs) that model nonlinear physical systems, and many papers have been written on the problem: for example, near-singular solutions of the Navier-Stokes equations and singular solutions of the Euler equations [5]; the possibility of singularities arising in Burger's equation [6]; some methods were presented for the investigation of the singularity formation in the NLS equation [7]; near-singular solutions of the complex Ginzberg-Landau equation from the viewpoint that the NLS equation is the conservation limit of the complex Ginzberg-Landau equations [8]; etc. However, most methods used in studying singular solutions are based on perturbation techniques or the singular point analysis. Also, the explicit forms of singular solution are seldom discussed, except for rational solutions. In this paper, we try to obtain the analytic singular solutions of nonlinear PDEs by two direct methods: an improved homogeneous balance (HB) method [9] and the invariant-Bäcklund transformation [10] based on a special nonlinear transformation.

The homogeneous balance method [11] has shown its efficiency in finding analytic solitary wave solutions of many PDEs. Its essence can be presented as follows: the nonlinear PDE is given by

$$P(u,u_x,u_t,u_{xx},u_{xt},u_{tt},\ldots)=0. \tag{1}$$

Supposing the solution of Eq. (1) is of the form

$$u=\sum_{i=0}^{N} a_i f^{(i)}(\varphi(x,t)), \tag{2}$$

where i is an integer, a_i are constant coefficients, $f(\varphi(x,t))$ is a function of $\varphi(x,t)$, $\varphi(x,t)$ is a function of x and t, and superscript (i) represents the derivative index. According to the assumption of homogeneous balance, in the PDE (1), the nonlinear terms and the highest-order partial derivative terms ought to be partially balanced. Then N is obtained, the expression of $f(\varphi(x,t))$ and the relation of $f^{(i)}\cdot f^{(j)}$ and $f^{(i+j)}$ can be derived. Assuming $\varphi(x,t)=b\ln(1+e^{\alpha x+\beta t+\gamma})$ and substituting formula (2) into Eq. (1), after deciding coefficients $a_i,b,\alpha,\beta,\gamma$, the solitary wave solution of the nonlinear PDE is obtained. Recently, an improved HB method [9] has been reported: after balancing the nonlinear and the highest-order partial derivative terms, substituting $f(\varphi(x,t))$ and the relation of $f^{(i)}\cdot f^{(j)}$ and $f^{(i+j)}$ into Eq. (1), we get

$$F(f',f'',\ldots,\varphi_x,\varphi_{xx},\ldots,\varphi_t,\varphi_{tt},\ldots,\varphi_{xt},\ldots)=0,$$

where F is a function of $f',f'',\ldots,\varphi_x$, $\varphi_{xx},\ldots,\varphi_t,\varphi_{tt},\ldots,\varphi_{xt},\ldots$. Obviously, F is a linear polynomial of $f',f'',\ldots$. Setting the coefficients of $f',f'',\ldots$ to zero yields a set of partial differential equations of $\varphi(x,t)$. Taking note of the terms of f', their forms must be $f'\varphi_{x_1,x_2,\ldots,x_p,t_1,\ldots,t_q}$. So the coefficient of f' in $F(f',f'',\ldots,\varphi_x,\varphi_{xx},\ldots,\varphi_t,\varphi_{tt},\ldots,\varphi_{xt},\ldots)=0$ is a linear polynomial $\sum_{i=1}^{k}a_i\varphi_{x_1,x_2,\ldots,x_{pi},t_1,\ldots,t_{qi}})$, and then the set of partial differential equations has an important feature, i.e., the equation from the coefficients of f' is a linear PDE of $\varphi(x,t)$, while the other equations can be regarded as constraint conditions to the linear equation. Thus solving the set of PDEs reduces to solving the linear PDE with some constraint conditions, and then the analytic solution (including traveling and nontraveling wave solutions) of nonlinear PDEs can be obtained by the improved HB method.

As an example, the Boussinesq equation and the Korteweg–de Vries (KdV) equation (fundamental models of

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the shallow water wave problem) are discussed. Analytic singular solutions of the equations are obtained by the improved homogeneous balance (HB) method. In particular, a finite-time singular solution of the Boussinesq equation is obtained in this paper, which was produced from a nonsingular physical field in the process of time evolution. The finite-time singular solution is relevant to a conjecture—in hydrodynamics many authors have shown that singularities of fluid motion can be formed in finite time, and some simplified models of fluid motion have been presented to demonstrate the conjecture [12]. Here, the Boussinesq equation gives another simplified model of fluid flow to do the same job, since this equation is a fundamental model of the shallow water wave problem.

The Bäcklund transformations were developed in the 1880s for use in the related theories of differential geometry and differential equations. Afterward, the relationships between the Bäcklund transformations and the inverse scattering transform [13] or the bilinear form [14] were presented, and the Bäcklund transformations have been used to find analytic solutions of nonlinear PDEs. In this paper, based on a special nonlinear transformation, the invariant-Bäcklund transformation of a generalized KdV equation is obtained and the analytic singular solution of it is obtained.

This paper is organized as follows. In Sec. II, an analytic singular solution of the Boussinesq equation is obtained by the improved HB method, which shows that the finite-time and the infinite-time singularity both exist in the Boussinesq equation. In Sec. III, an analytic singular solution of the KdV equation is obtained by the improved HB method and an analytic periodic singular solution in space of the generalized KdV equation is obtained by the invariant-Bäcklund transformation. In Sec. IV, the main results are given.

II. BOUSSINESQ EQUATION

The Boussinesq equation [15] is a classical model of long wavelength hydrodynamic waves and other physical systems and is written in the form

$$u_{tt}-u_{xx}-a(u^2)_{xx}+bu_{xxxx}=0. \tag{3}$$

where a, b are real constants. It can be derived from the incompressible fluid equations

$$\begin{aligned}\partial_t u&=-\frac{1}{\rho}\partial_x p',\\ \partial_t w&=-\frac{1}{\rho}\partial_z p',\\ \partial_x u+\partial_z w&=0,\end{aligned} \tag{4}$$

where u, w are velocities in the x, z directions, and p' is the relative pressure. Equation (3) can be derived from the Toda lattice also. The Boussinesq equation (for wave propagation in both directions) describes wave motion in weakly nonlinear and dispersive media, and a suitable approximation enables the KdV equation (for which wave propagation is restricted to one direction) to be derived from this equation. Much work has been reported for this equation, for example, the soliton and the Painlevé expansion [16], the soliton and the bilinear form (Hirota's method) [17], periodic wave solutions and the structure of the rational solution to Boussinesq equation [18], etc.

Here, the improved HB method is employed to obtain the singular solution of the Boussinesq equation. Supposing the solution is of the form

$$u=\sum_{i=0}^{N} f^{(i)}(\omega(x,t)), \tag{5}$$

where N is an integer, substituting formula (5) into Eq. (3), and balancing the nonlinear term $a(u^2)_{xx}$ and the linear term bu_{xxxx}, we get $N=2$, and then

$$u=f''\omega_x^2+f'\omega_{xx}. \tag{6}$$

Substituting formula (6) into Eq. (3), and collecting all homogeneous terms in partial derivatives of $\omega(x,t)$, we have

$$\begin{aligned}&[bf^{(6)}-2af'''f'''-2af''f^{(4)}]\omega_x^6+[15bf^{(5)}-24af''f'''-2af'f^{(4)}]\omega_x^4\omega_{xx}\\&+[f^{(4)}\omega_t^2\omega_x^2-f^{(4)}\omega_x^4+(45bf^{(4)}-24af''f''-12af'f''')\omega_x^2\omega_{xx}^2+(20bf^{(4)}-8af''f''-4af'f''')\omega_x^3\omega_{xxx}]\\&+[(\omega_{tt}\omega_x^2+\omega_t\omega_x\omega_{xt}+\omega_t^2\omega_{xx}-6\omega_x^2\omega_{xx})f'''+(15bf'''-6af'f'')\omega_{xx}^3+(60bf'''-20af'f'')\omega_x\omega_{xx}\omega_{xxx}\\&+(15bf'''-2af'f'')\omega_x^2\omega_{xxxx}]+[(2\omega_{xt}^2+2\omega_x\omega_{xtt}+\omega_{tt}\omega_{xx}-3\omega_{xx}^2+2\omega_t\omega_{xxt}-4\omega_x\omega_{xxx})f''+(10bf''-2af'f')\omega_{xxx}^2\\&+(15bf''-2af'f')\omega_{xx}\omega_{xxxx}+6bf''\omega_x\omega_{xxxxx}]+(\omega_{xxtt}+b\omega_{xxxxxx}-\omega_{xxxx})f'=0.\end{aligned} \tag{7}$$

Setting the coefficient of ω_x^6 in Eq. (7) to zero yields an ordinary differential equation for f, namely,

$$bf^{(6)}-2af'''f'''-2af''f^{(4)}=0. \tag{8}$$

The solution of Eq. (8) is obtained as

$$f=-\frac{6b}{a}\ln\omega, \tag{9}$$

which yields

$$f''f'''=\frac{b}{2a}f^{(5)},\quad f'f^{(4)}=\frac{3b}{2a}f^{(5)},\quad f''f''=\frac{b}{a}f^{(4)},$$
$$f'f'''=\frac{2b}{a}f^{(4)},\quad f'f''=\frac{3b}{a}f''',\quad f'f'=\frac{6b}{a}f''. \tag{10}$$

Substituting formulas (10) into Eq. (7), it can be simplified to a linear polynomial of $f',f'',\ldots$; then setting the coefficients of $f',f'',\ldots$ to zero yields a set of partial differential equations for $\omega(x,t)$,

$$\omega_t^2-\omega_x^2-3b\,\omega_{xx}^2+4b\,\omega_x\omega_{xxx}=0, \tag{11}$$

$$\omega_{tt}\omega_x^2+4\omega_t\omega_x\omega_{xt}+\omega_t^2\omega_{xx}-6\omega_x^2\omega_{xx}-3b\,\omega_{xx}^3+9b\,\omega_x^2\omega_{xxxx}=0, \tag{12}$$

$$2\omega_{xt}^2+2\omega_x\omega_{xtt}+\omega_{tt}\omega_{xx}-3\omega_{xx}^2+2\omega_t\omega_{xxt}-4\omega_x\omega_{xxx}-2b\,\omega_{xxx}^2+3b\,\omega_{xx}\omega_{xxxx}+6b\,\omega_x\omega_{xxxxx}=0, \tag{13}$$

$$\omega_{xxtt}+b\,\omega_{xxxxxx}-\omega_{xxxx}=0. \tag{14}$$

We note that the Eq. (14) from the coefficients of f' is a linear PDE for $\omega(x,t)$. Then Eqs. (11)–(13) can be regarded as constraint conditions to the linear equation (14); thus solving Eq. (3) reduces to solving the linear PDE (14) with some constraint conditions (11)–(13).

Equation (14) may be integrated once to yield

$$\omega_{xtt}+b\,\omega_{xxxxx}-\omega_{xxx}=p(t), \tag{15}$$

where $p(t)$ is an arbitrary function of time. It is easy to know that Eq. (14) has the solution

$$\omega(x,t)=S(\xi)+q(t), \tag{16}$$

where $S(\xi)$ is the traveling wave solution of equation $\omega_{xtt}+b\,\omega_{xxxxx}-\omega_{xxx}=0$, and $q(t)$ satisfies equation $(d^2/dt^2)(q(t))=p(t)$. So the solution of Eq. (14) is of the form

$$\omega(x,t)=d_0+d_1(x-vt)+d_2(x-vt)^2+d_3e^{\beta(x-vt)}+d_4e^{-\beta(x-vt)}+q(t), \tag{17}$$

where $\beta=\sqrt{(1-v^3)/b}>0$; note that solution (17) is a nontraveling wave solution of Eq. (14). Substituting formula (17) into the constraint conditions (11)–(13), a set of ordinary differential equations are obtained. Solving this set of equations, we find

$$d_2=0,$$
$$d_3=\frac{bd_1^2(-1+4v^2)}{12d_4v^2(-1+v^2)},$$
$$q(t)=-\frac{d_1(-1+v^2)}{v}t. \tag{18}$$

Thus the solutions of Eqs. (11)–(14) are obtained as

$$\omega(x,t)=d_0+d_1(x-vt)+\frac{bd_1^2(-1+4v^2)}{12d_4v^2(-1+v^2)}e^{\beta(x-vt)}+d_4e^{-\beta(x-vt)}\frac{d_1(-1+v^2)}{v}t, \tag{19}$$

where b, v, d_0, d_1, and d_4 are arbitrary constants. Substituting solution (19) into formulas (9) and (6), the analytic solution of Eq. (3) is obtained, namely

$$u(x,t)=\frac{6b\left(d_1-d_4e^{-\beta(x-vt)}\beta+\dfrac{bd_1^2(-1+4v^2)\beta e^{\beta(x-vt)}}{12d_4v^2(-1+v^2)}\right)^2}{a\left[d_0+d_4e^{-\beta(x-vt)}-\dfrac{d_1}{v}(-1+v^2)t+\dfrac{bd_1^2(-1+4v^2)e^{\beta(x-vt)}}{12d_4v^2(-1+v^2)}+d_1(x-vt)\right]^2}-\frac{6b\left(\dfrac{d_4e^{-\beta(x-vt)}(1-v^2)}{b}+\dfrac{d_1^2(1-v^2)(-1+4v^2)e^{\beta(x-vt)}}{12d_4v^2(-1+v^2)}\right)}{a\left[d_0+d_4e^{-\beta(x-vt)}-\dfrac{d_1(-1+v^2)t}{v}+\dfrac{bd_1^2(-1+4v^2)e^{\beta(x-vt)}}{12d_4v^2(-1+v^2)}+d_1(x-vt)\right]}, \tag{20}$$

where a, b, v, d_0, d_1, and d_4 are arbitrary constants. Here a just is a parameter to decide the relative value of $u(x,t)$, and when $d_1=0$, the solution (20) decays to the solitary wave solution.

Solution (20) is very complex and includes six arbitrary constants, so it needs discussion. It's easy to understand that the singularity of solution (20) comes from the zero value of formula (17), namely, it is decided by $\omega(x,t)=0$. If the analytic solution $x=T(t)$ or $t=X(x)$ of $\omega(x,t)=0$ is obtainable, the main property of the solution can be given, but formula (17) includes the transcendental function, so it's impossible to get an explicit solution. By numerical analytics, the main property of the solution is reported as follows: when $0.5>v$ or $v<-0.5$, $b<0$, $d_1<0$, $d_4<0$, the unlimited-time blowup solution exists, and when $0.5\geqslant v\geqslant -0.5(v\neq 0)$, $b>0$, $d_1>0$, $d_4>0$, the finite-time blowup solution exists.

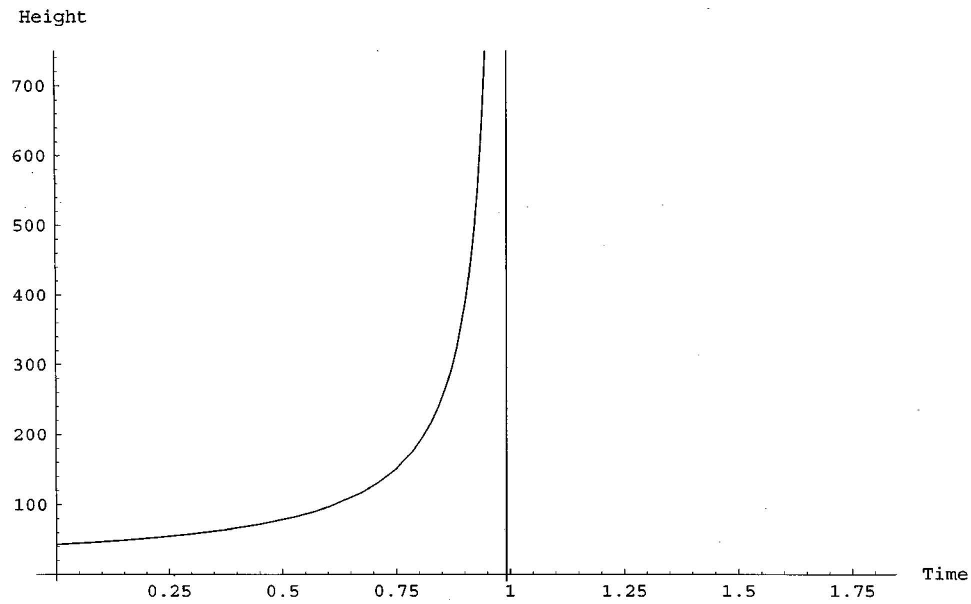

FIG. 1. A finite-time blowup evolution is shown of a finite-time singular solution; the values of the parameters are as follows: $a=-0.1$, $b=1$, $v=-0.5$, $d_0=0.5$, $d_1=1$, $d_2=1$. In the figure, the max value $u(x,t)$ and the asymptotic blowup time is indicated.

Figure 1 shows a finite-time blowup solution. Note that quantities plotted in all figures are dimensionless. The numerical results suggest that: (1) the constant v decides the region of the singularity, when $0.5 \geqslant v > 0$, the singularity exists in the half-line $[-\infty, t_0]$ and when $0 > v \geqslant -0.5$, the singularity exists in the half-line $[t_0, \infty]$; (2) the constant a just decides the relative values of $u(x,t)$, and the constants b, d_0, d_1, and d_4 decide the values of t_0, which is the blowup time; and (3) there are two kinds of finite-time blowup evolution modes at $v=-0.5$ or $v=0.5$ and $0.5 > v > -0.5$. Figure 2 ($v=-0.5$) shows a time evolution mode of the finite-time blowup solution. Figure 3 ($0 > v > -0.5$) shows another time evolution mode of the finite-time blowup solution.

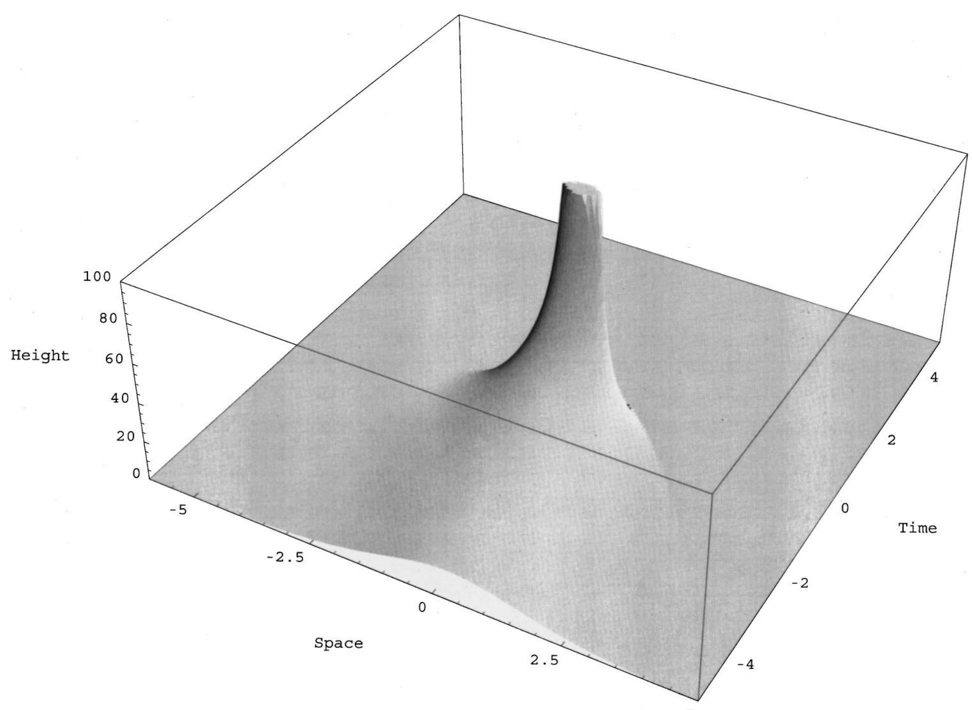

FIG. 2. A finite-time blowup solution is shown; the values of the parameters are as follows: $a=-0.1$, $b=1$, $v=-0.5$, $d_0=0.5$, $d_1=1$, $d_2=1$. At $t=0$, $u(x,t)$ looks like a hump (a standard solitary wave), the height of the hump becomes higher and higher, and at $t \simeq 1.0$, the height of the hump tends to infinity.

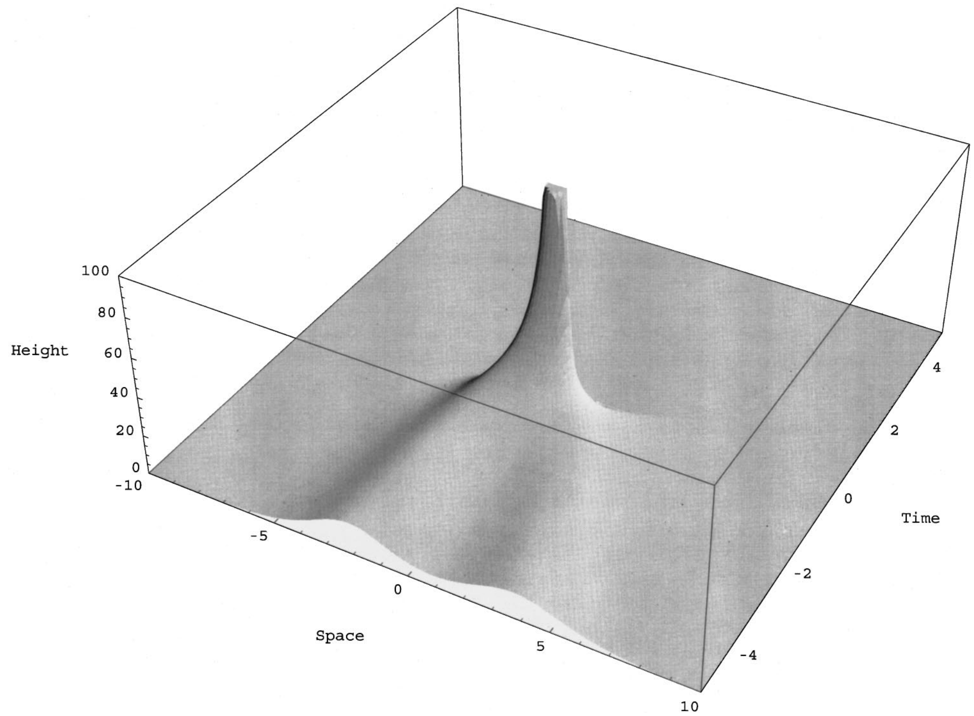

FIG. 3. A finite-time blowup solution is shown; the values of the parameters are as follows: $a=-0.1$, $b=1$, $v=-0.2$, $d_0=0.5$, $d_1=1$, $d_2=1$. At $t=0$, $u(x,t)$ looks like two humps, one hump runs to another hump, at $t\simeq 1.0$, two humps come into collision and become a single hump, after which the height of the hump tends to infinity.

III. KdV EQUATION AND GKdV EQUATION

As is known, the KdV equation is not only of mathematical interest but also of practical importance. It has been shown to describe small amplitude shallow water waves, hydromagnetic waves in a cold plasma, ion-acoustic waves, acoustic waves in an anharmonic crystal, and wave motions in other biological and physical systems. The standard KdV equation is given by

$$u_t+\alpha u u_x+\beta u_{xxx}=0, \tag{21}$$

where α,β are real constants. When the improved HB method is applied to the KdV equation, the following analytic solution of the KdV equation is obtained:

$$\begin{aligned} u(x,t)=&(48d_4e^{b(x-vt)}v\{3\beta^2d1^3e^{2b(x-vt)}b\\ &+\beta d_1v[-d_1e^{b(x-vt)}(8d_4+d_0e^{b(x-vt)})\\ &+8d_4^2b+d_1^2e^{2b(x-vt)}(3vt-x)]\\ &+4d_4^2[d_0+d_1(x-3vt)]\})/[a(\beta d_1^2e^{2b(x-vt)}\\ &-4d_4v\{d_4+e^{b(x-vt)}[d_0+d_1(x-3vt)]\})^2]. \end{aligned} \tag{22}$$

It is easy to show that solution (22) is an unlimited-time singular solution.

Recently, some generalized KdV equations (GKdV) have been discussed widely, including a few high-order KdV equations [19], the q KdV equation [20], and some generalized KdV equations [21]. One generalized KdV [10]

$$u_t+(n+1)(n+2)u^nu_x+u_{xxx}=0, \tag{23}$$

has been presented and a more generalized form

$$u_t+a(u^m)_x+b(u^l)_{xxx}=0, \tag{24}$$

has been introduced by Rosenau and Hyman [22]. Equation (24) has yielded the compactons solution (23) for certain values of m and l, which, like solitons, have the remarkable property that after colliding with other compactons, they reemerge in the same coherent shape. In the paper, a concrete equation of the GKdV Equation (23) (in the case $n=3$)

$$u_t+20u^3u_x+u_{xxx}=0, \tag{25}$$

is considered. For a discussion of Eq. (25), the nonlinear transformation

$$u(x,t)=\frac{g(x,t)}{f(x,t)^{2/3}}+u_0(x,t), \tag{26}$$

is introduced. Then, substituting solution form (26) into Eq. (25), and making the coefficients of like powers of $f(x,t)$ vanish, we obtain the following set of equations,

$$
\begin{aligned}
&-\tfrac{40}{27}gf_x(9g^3+2f_x^2)=0 \quad \text{for } f^{-11/3},\\
&-40g^3u_0f_x=0 \quad \text{for } f^{-9/3},\\
&\tfrac{10}{3}(6g^3g_x+f_x^2g_x+gf_xf_{xx})=0 \quad \text{for } f^{-8/3},\\
&-40g^2u_0^2f_x=0 \quad \text{for } f^{-7/3},\\
&20g^2(3u_0g_x+gu_{0x})=0 \quad \text{for } f^{-6/3},\\
&-\tfrac{2}{3}(gf_t+20gu_0^3f_x+3g_xf_{xx}+3f_xg_{xx}+gf_{xxx})\\
&\quad=0 \quad \text{for } f^{-5/3},\\
&60gu_0(u_0g_x+gu_{0x})=0 \quad \text{for } f^{-4/3},\\
&g_t+20u_0^3g_x+60gu_0^2u_{0x}+g_{xxx}=0 \quad \text{for } f^{-2/3},
\end{aligned}
\tag{27}
$$

where $u_0(x,t)$ satisfies the equation

$$u_{0t}+20u_0^3u_{0x}+u_{0xxx}=0, \tag{28}$$

which is the same as Eq. (25) for $u(x,t)$; thus the set of Eqs. (26)–(28) constitutes an invariant-Bäcklund transformation of the GKdV Eq. (25).

It is obvious that Eq. (28) has the trivial solution $u_0(x,t)=0$, then substituting $u_0(x,t)=0$ into formula (26), the transformation is simply

$$u=\frac{g(x,t)}{f(x,t)^{2/3}} \tag{29}$$

and Eq. (27) become

$$
\begin{aligned}
&g_t+g_{xxx}=0,\\
&gf_t+3g_xf_{xx}+3f_xg_{xx}+gf_{xxx}=0,\\
&6g^3g_x+f_x^2g_x+gf_xf_{xx}=0,\\
&9g^3+2f_x^2=0.
\end{aligned}
\tag{30}
$$

The first and last equations are easily solved (the other equations can be treated as constraint conditions), from which solutions of Eq. (30) can be obtained. Substituting the solutions into transformation (29), the analytic solution of the GKdV Eq. (25) is obtained as

$$u(x,t)=-\frac{v^{1/3}\sec^{4/3}\left(\dfrac{-16c+9v(x+vt)}{12\sqrt{v}}\right)}{2\tan^{2/3}\left(\dfrac{-16c+9v(x+vt)}{12\sqrt{v}}\right)}, \tag{31}$$

where c is a constant, and v is the velocity of the traveling wave. It's easy to understand that the singularity of solution (31) is decided by equation $\tan^{2/3}\{[-16c+9v(x+vt)]/12\sqrt{v}\}=0$. Thus the solution (31) is a space periodic singular solution, i.e., the GKdV equation contains the unlimited-time singular solution, which is periodic in space.

IV. CONCLUSION

In conclusion, two methods to seek analytic solutions of nonlinear PDEs are introduced: an improved HB method and the invariant-Bäcklund transformation based on a special nonlinear transformation, and two kinds of analytic singular solutions (finite-time and infinite-time singular solutions) of the Boussinesq equation and a GKdV equation (including the KdV equation) are obtained. A finite-time singular solution of the Boussinesq equation is discussed, which is relevant to singularity formation in finite time in hydrodynamics. A periodic in space unlimited-time singular solution of the GKdV equation is obtained, and the invariant-Bäcklund transformation of the equation is presented. In addition, the nonlinear transformation $u=(g/f_n^2)+u_0$ is useful in finding solutions of Eq. (23).

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